Difference Schemes Derived from the Integral Theorems of Fluid Mechanics

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Theme

A FAMILY of difference schemes is derived for the calculation of unsteady, compressible flow of an inviscid gas. The schemes can be applied to an arbitrary mesh geometry, and the mesh is allowed to deform in time. Therefore, the mesh can be chosen to coincide with an arbitrary stationary or nonstationary boundary. Also, the schemes can be used to fit shockwaves and contact surfaces by allowing the mesh to deform in concert with these discontinuities. Alternatively, the schemes allow shocks to be calculated by the shock-capturing technique.

Recently, several authors[1-3] have used a MacCormack scheme, modified for application to arbitrary mesh geometries by using the integral theorems. Schiff[4] extended the method to allow for time-varying meshes. These schemes, however, suffer from low accuracy if applied to non-Cartesian meshes. Even in the relatively simple case of one-dimensional, unsteady flow with nonuniform mesh, these schemes have a truncation error of the first order.

By contrast, the schemes described in this paper are second-order accurate in the two-dimensional and first-order accurate in the three-dimensional case.

The two-step schemes discussed here are closely related to the Lax-Wendroff scheme and reduce to it for stationary Cartesian grids.

Content

The schemes are derived from the integral theorems which are formulated in conservation form for a control volume in space-time coordinates as:

$$\int (U_t + F_x + G_y)\,\mathrm{d}x\mathrm{d}y\,\mathrm{d}t = 0 \tag{1}$$

Here the vectors U, F, and G are defined in the usual way as:

$$U = (\rho, \rho u, \rho v, E)$$

$$F = (\rho u, \rho u^2 + P, \rho uv, u(E+P))$$

$$G = (\rho v, \rho uv, \rho v^2 + P, v(E+P)$$

Equation (1) is supplemented by the equation of state for a perfect gas

$$E = P/(\gamma - 1) + \rho(u^2 + v^2)/2$$

In the formulation of Eq. (1), the momentum equation is replaced by the impulse momentum theorem and the rate of energy equation by the energy equation.

The difference schemes are now derived as follows: A control volume geometry is assumed and the values of U, F, and G are approximated within this volume by standard finite-element methods. The integrals are then evaluated with the same techniques. If the control volume is chosen as a pyramid such that it contains only one point in the $t+\Delta t$ time level and all others in the initial data plane t, the resulting difference scheme is explicit in nature. Only this case is considered here.

The only geometries considered in this paper are the triangle in two dimensions and the tetrahedron in three dimensions. More complicated control volumes are subdivided into these basic elements.

We now present the results for these geometries separately for the case of two and three dimensions.

Two-Dimensional Schemes

For the triangles of Fig. 1 we denote the properties of U and F at the vertices with the corresponding subscript. The difference scheme for the single triangle in Fig. 1a is:

$$hU_3 = (h-p)U_1 + pU_2 - \Delta t(F_2 - F_1) \tag{2}$$

For the element in Fig. 1b we obtain the two-step difference scheme

$$h_1 U_4 = \ell_1 U_1 + (h_1 - \ell_1)U_2 - \sigma\Delta t(F_2 - F_1)$$

$$h_2 U_5 = (h_2 - \ell_2)U_2 + \ell_2 U_3 - \sigma\Delta t(F_3 - F_2) \tag{3}$$

$$(\ell_1 + \ell_2)U_6 = (\ell_1 + \ell_2)U_2 + p(U_5 - U_4) - \Delta t(F_5 - F_4)$$

Here the quantities ℓ_1 and ℓ_2 are obtained as indicated in the figure

$$\ell_1 = \sigma(h_2 - p) \qquad \ell_2 = \sigma(h_1 + p)$$

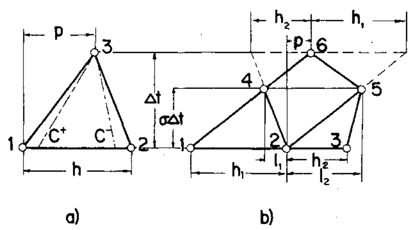

Fig. 1 Two-dimensional elements.

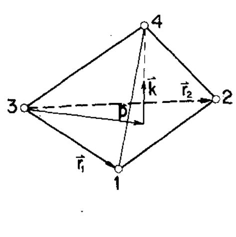

Fig. 2 The basic tetrahedron.

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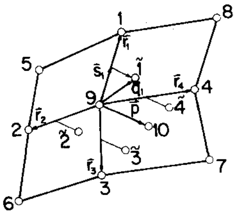

Fig. 3 Three-dimensional element.

The scheme of Eq. (2) is first-order accurate while that of Eq. (3) is second-order accurate. Both schemes are stable if the CFL-condition is satisfied, i.e., if the characteristics C^+ and C^- intersect the initial data line inside the intervals [1,2] and [1,3], respectively. Notice that, if both characteristics have positive slope and if $p>h$ in Fig. 1a, then the allowable time step is bracketed by $\Delta t_{\min} \leq \Delta t \leq \Delta t_{\max}$. Similar observations apply to the scheme in Fig. 1b. This situation can arise at boundary points or where shocks or contact surfaces are fitted. In this case a quasi-implicit scheme described in the report is used to advantage.

The scheme Eq. (3) reduces to the Lax-Wendroff scheme if $p=0$, $\sigma=1/2$, and $h_1=h_2$.

Three-Dimensional Schemes

For the tetrahedron in Fig. 2 the points 1, 2, and 3 lie in the initial data plane. We seek the data at the point 4 in the $(t+\Delta t)$ plane, whose projection onto the initial data plane is characterized by the vector $\boldsymbol{p}$. We introduce the unit vector $\boldsymbol{k}$ normal to the initial data plane and the vector $\boldsymbol{H}=(F, G, 0)$ for each component of U. Also, we denote the triple scalar product by $\boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{c}=[\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}]$. Then the difference scheme for each component of U is written in the form

$$[r_1,r_2,k](U_4-U_3)=[p,r_2,k]U_1-[p,r_1,k]U_2-[p,r_2-r_1,k]U_3$$
$$+\Delta t\{[r_2,H_1,k]-[r_1,H_2,k]-[r_2-r_1,H_3,k]\} \qquad (4)$$

This scheme is first-order accurate and its stability is governed by the CFL-condition.

The top view of a more elaborate element is shown in Fig. 3. Points 1 through 9 lie in the initial data plane, point 10 in the plane $(t+\Delta t)$. Points $\tilde{1}$ through $\tilde{4}$ are auxiliary points at $(t+\Delta t/2)$. Their projection onto the initial data plane, q_i, are given by $q_i=s_i+p/2$. The vectors s_i are determined from

$$s_i=\lambda_i r_i, \quad \lambda_i=|r_{i\pm 2}|/2|r_i| \qquad (i=1,2,3,4)$$

where r_i is the vector from point 9 to the point i in the initial data plane and the sign is chosen such that $1\leq i\pm 2\leq 4$. Next we introduce the auxiliary quantities ρ_i, X_i, and K_i which are defined for $\tilde{1}$ as follows

$$\rho_1=r_2+r_5-r_4-r_8$$

$$X_1=U_2+U_5-U_4-U_8$$

$$K_1=H_2+H_5-H_4-H_8$$

For the points 2 to 4 these properties are obtained in similar fashion by rotating Fig. 3.

Then the component equation for the properties at the auxiliary points, denoted by a tilde, is

$$[r_i,\rho_i,k](\tilde{U}_i-U_9)=[q_i,\rho_i,k](U_i-U_9)$$
$$-[q_i,r_i,k]X_i+\frac{\Delta t}{2}\{[\rho_i,H_i-H_9,k]-[r_i,K_i,k]\} \qquad (5a)$$

For the second step we obtain

$$[\ell,m,k](U_{10}-U_9)=[p,m,k](\tilde{U}_1-\tilde{U}_3)$$
$$+[p,\ell,k](\tilde{U}_4-\tilde{U}_2)+\Delta t\{[m,\tilde{H}_1-\tilde{H}_3,k]+[\ell,\tilde{H}_4-\tilde{H}_2,k]\} \qquad (5b)$$

where

$$\ell=q_1-q_3=s_1-s_3 \qquad m=q_2-q_4=s_2-s_4$$

Again, for a Cartesian mesh and with $p=0$, the scheme reduces to the Lax-Wendroff scheme. The scheme is first-order accurate for all geometries and second-order accurate for the special case of a parallelogram grid. The stability condition is similar to that for the Lax-Wendroff scheme: The backward Mach cone through point 10 must intersect the initial data plane inside the polygon similar to that spanned by the points 1 to 8 but reduced in size by a factor $1/\sqrt{8}$. For boundary points or in the fitting of discontinuities, a similar problem as described in the two-dimensional case may occur. It has been observed that the use of the simpler scheme, Eq. (4) with its less stringent stability criterion, successfully avoided instability without loss of accuracy.

The application of boundary conditions on a moving surface and the shock-fitting technique are briefly described in the paper.

Acknowledgment

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